GAMES IN BANACH SPACES

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Abstract. The notion of Aronszajn-null sets generalizes the notion of Lebesgue measure zero on the real line and the Euclidean space to infinite dimensional Banach spaces. We present a game-theoretic approach to Aronszajn-null sets, establish its basic properties, and discuss the ensuing open problems.

1. Motivation

Aronszajn null sets were introduced by Aronszajn in the context of studying a.e. differentiability of Lipschitz mappings between Banach spaces. Christensen, Phelps and Mankiewicz studied the same problem independently and used Haar null, Gaussian null and cube null sets, respectively. See the monograph [1] for more information about the history. Csörnyei [2] proved that Aronszajn null, Gaussian null, and cube null sets coincide. It is well known that Haar null sets form a strictly larger family than Aronszajn null sets (see [1]).

One of the questions in the differentiability theory is to understand the structure of the sets of points of Gâteaux nondifferentiability of Lipschitz mappings defined on separable Banach spaces. The strongest result in this context is due to Preiss and Zajíček [3]. Let $\mathcal{A}$ denote the family of (Borel) Aronszajn-null sets (to be defined in the sequel). Preiss and Zajíček introduced a Borel σ-ideal $\tilde{\mathcal{A}}$ such that $\tilde{\mathcal{A}} \subseteq \mathcal{A}$. It follows from a recent result of Preiss that $\mathcal{A} = \tilde{\mathcal{A}}$ in $\mathbb{R}^2$, and it is unknown for $2 < \dim X < \infty$. In infinite dimensions, the inclusion $\tilde{\mathcal{A}} \subseteq \mathcal{A}$ is strict. It is also unknown whether, according to the definitions of [3], $\tilde{\mathcal{A}} = \tilde{\mathcal{C}}$ and $\tilde{\mathcal{C}} \subseteq \mathcal{A}^*$.

Understanding the structure of the sets of points of non-differentiability could possibly also be helpful in answering the longstanding open problem whether two separable Lipschitz isomorphic spaces are actually linearly isomorphic. This is known for some special Banach spaces, but is open for example for ℓ_1 and L_1.

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We introduce a game-theoretic approach to Aronszajn null sets. It would be interesting to see whether this new perspective can yield interesting results which do not involve the new notions. One idea behind this approach is that the Aronszajn null sets are defined as sets for which there exists a certain decomposition for each complete sequence of directions, whereas in the game setting, such a decomposition is being constructed while we are only given one direction at a time.

2. The Aronszajn-null game

Let X be a separable Banach space (over $\mathbb{R}$). The following definitions are classical:

(1) For a nonzero $x \in X$, $\mathcal{A}(x)$ denotes the collection of all Borel sets $A \subseteq X$ such that for each $y \in X$, $A \cap (\mathbb{R}x + y)$ has Lebesgue (one dimensional) measure zero.
(2) A Borel set $A \subseteq X$ is *Aronszajn-null* if for each dense sequence $\{x_n\}_{n\in\mathbb{N}} \subseteq X$, there exist elements $A_n \in \mathcal{A}(x_n)$, $n \in \mathbb{N}$, such that $A \subseteq \bigcup_n A_n$.
(3) $\mathcal{A}$ denotes the collection of Aronszajn-null sets.

$\mathcal{A}$ is a Borel σ-ideal.

Remark 2.1. Replacing "dense" by "complete" in item 2 of the above definition of Aronszajn-null sets, one gets an equivalent definition [1, Corollary 6.30].

The definition of Aronszajn-null sets motivates the following.

Definition 2.2. The *Aronszajn-null game* $\mathcal{A}_{\mathsf{G}}$ for a Borel set $A \subseteq X$ is a game between two players, **I** and **II**, who play an inning per each natural number. In the nth inning, **I** picks $x_n \in X$, and **II** responds by picking $A_n \in \mathcal{A}(x_n)$. This is illustrated in the following figure.

$$\begin{array}{lcccccccc} \mathbf{I}: & x_1 \in X & & & & x_2 \in X & & & \dots \\ & & \searrow & & \nearrow & & \searrow & & \\ \mathbf{II}: & & & A_1 \in \mathcal{A}(x_1) & & & & A_2 \in \mathcal{A}(x_2) & \dots \end{array}$$

I is required to play such that $\{x_n\}_{n\in\mathbb{N}}$ is dense in X. **II** wins the game if $A \subseteq \bigcup_n A_n$; otherwise **I** wins.

For a game G, the notation $\mathbf{I} \uparrow \mathsf{G}$ is a shorthand for "**I** has a winning strategy in the game G", and $\mathbf{I} \not\uparrow \mathsf{G}$ stands for "**I** does not have a winning strategy in the game G". Define $\mathbf{II} \uparrow \mathsf{G}$ and $\mathbf{II} \not\uparrow \mathsf{G}$ similarly. The following is easy to see.

Lemma 2.3. *If* $\mathbf{I} \not\uparrow \mathcal{A}_{\mathsf{G}}$ *for* A*, then* A *is Aronszajn-null.* □

The converse is open.

Conjecture 2.4. *If A is Aronszajn-null, then* $\mathbf{I} \not\uparrow \mathcal{A}_{\mathsf{G}}$ *for A.*

Lemma 2.5. *The property* $\mathbf{II} \uparrow \mathcal{A}_{\mathsf{G}}$ *is preserved under taking Borel subsets and countable unions, i.e., it defines a Borel σ-ideal.*

Proof. It is obvious that $\mathbf{II} \uparrow \mathcal{A}_{\mathsf{G}}$ is preserved under taking Borel subsets. To see the remaining assertion, assume that $B_1, B_2, \dots$ all satisfy $\mathbf{II} \uparrow \mathcal{A}_{\mathsf{G}}$, and for each k let F_k be a winning strategy for **II** in the game $\mathcal{A}_{\mathsf{G}}$ played on B_k. Define a strategy F for **II** in the game $\mathcal{A}_{\mathsf{G}}$ played on $\bigcup_k B_k$ as follows. Assume that **I** played $x_1 \in X$ in the first inning. For each k let $A_{k,1} = F_k(x_1)$, and set $A_1 = \bigcup_k A_{k,1} \in \mathcal{A}(x_1)$. **II** plays A_1. In the nth inning we have $(x_1, A_1, x_2, A_2, \dots, x_n)$ given, where x_n is the nth move of **I**. For each k let $A_{k,n} = F_k(x_1, A_{k,1}, x_2, A_{k,2}, \dots, x_n)$, and set $A_n = \bigcup_n A_{k,n} \in \mathcal{A}(x_n)$. **II** plays A_n.

Consider the play $(x_1, A_1, x_2, A_2, \dots)$. For each k, $(x_1, A_{k,1}, x_2, A_{k,2}, \dots)$ is a play according to the strategy F_k, and therefore $B_k \subseteq \bigcup_n A_{k,n}$. Consequently,

$$B = \bigcup_{k\in\mathbb{N}} B_k \subseteq \bigcup_{k\in\mathbb{N}} \bigcup_{n\in\mathbb{N}} A_{k,n} = \bigcup_{n\in\mathbb{N}} \bigcup_{k\in\mathbb{N}} A_{k,n} = \bigcup_{n\in\mathbb{N}} A_n,$$

thus **II** won the play. □

A Borel set $A \subseteq X$ is *directionally-porous* if there exist $\lambda > 0$ and a nonzero $v \in X$ such that for each $a \in A$ and each positive ϵ, there is $x \in \mathbb{R}v + a$ such that $\|x - a\| < \epsilon$ and $A \cap B(x, \lambda\|x - a\|) = \emptyset$. If A is directionally-porous, then so is $\overline{A}$. A is *σ-directionally-porous* if it is a countable union of directionally-porous sets.

Proposition 2.6. *For each σ-directionally-porous set,* $\mathbf{II} \uparrow \mathcal{A}_{\mathsf{G}}$.

Proof. By Lemma 2.5, it suffices to consider the case where $A \subseteq X$ is directionally-porous. Let $\lambda > 0$ and $v \in X$ be witnesses for that. In this case, the function

$$F(x_1, A_1, x_2, A_2, \dots, x_n) = \begin{cases} A & \|x_n - v\| < \lambda/2 \\ \emptyset & \text{otherwise} \end{cases}$$

is a winning strategy for **II** in the game $\mathcal{A}_{\mathsf{G}}$. □

For a nonzero $x \in X$ and a positive ϵ, let $\mathcal{A}(x, \epsilon)$ denote the collection of all Borel sets $A \subseteq X$ such that for each $v \in X$ with $\|v - x\| < \epsilon$, $A \in \mathcal{A}(v)$. $\mathcal{C}^*$ is the collection of all countable unions of sets A_n such that each $A_n \in \mathcal{A}(x_n, \epsilon_n)$ for some x_n, ϵ_n. $\mathcal{C}^*$ is a Borel σ-ideal.

The proof of Proposition 2.6 actually establishes the following.

Proposition 2.7. *For each $A \in \mathcal{C}^*$, $\mathbf{II} \uparrow \mathcal{A}_{\mathsf{G}}$.* □

The following diagram summarizes our knowledge thus far:

$$\sigma\text{-directionally-porous} \Longrightarrow \mathcal{C}^* \Longrightarrow \mathbf{II} \uparrow \mathcal{A}_{\mathsf{G}} \Longrightarrow \mathbf{I} \not\uparrow \mathcal{A}_{\mathsf{G}} \Longrightarrow \mathcal{A}.$$

The open problems concerning this diagram are whether any of the last three arrows can be reversed (i.e., turned into an equivalence) and therefore produce a characterization. The first arrow is not reversible [3].

We conjecture that $\mathbf{II} \uparrow \mathcal{A}_{\mathsf{G}}$ is strictly stronger than $\mathcal{A}$. For shortness and clarity, we introduce the following.

Definition 2.8. For $Y \subseteq X$, $A \in \mathcal{A}(\wedge\, Y)$ means: For each $y \in Y$, $A \in \mathcal{A}(y)$. In other words,

$$\mathcal{A}(\wedge\, Y) = \bigcap_{y \in Y} \mathcal{A}(y).$$

Thus, $\mathcal{A}(x, \epsilon) = \mathcal{A}(\wedge\, B(x, \epsilon))$. Using this notation, we can see that the property $\mathbf{II} \uparrow \mathcal{A}_{\mathsf{G}}$ implies something quite close to $\mathcal{A}^*$, see Corollary 2.10.

Theorem 2.9. *Assume that $\mathbf{II} \uparrow \mathcal{A}_{\mathsf{G}}$ holds for A. Then: For each countable dense $D \subseteq X$ and each (pseudo)basis $\{U_n\}_{n\in\mathbb{N}}$ for the topology of X, there exist elements*

$$A_n \in \mathcal{A}(\wedge\, D \cap U_n),$$

$n \in \mathbb{N}$, such that $A \subseteq \bigcup_n A_n$.

Proof. Assume that $D \subseteq X$ is countable and dense, and $\{U_n\}_{n\in\mathbb{N}}$ is a (pseudo)basis $\{U_n\}_{n\in\mathbb{N}}$ for the topology of X. For each n, fix an enumeration $\{x_{n,m} : m \in \mathbb{N}\}$ of $D \cap U_n$.

Let F be a winning strategy for $\mathbf{II}$ in the game $\mathcal{A}_{\mathsf{G}}$. To each finite sequence η of natural numbers we associate a Borel set A_η and an element $y_\eta \in D \cap U_n$ where n is the length of the sequence. This is done by induction on n.
$n = 1$: For each k, set $A_k = F(x_{1,k})$.
$n = m + 1$: For each $\eta \in \mathbb{N}^m$ and each k, define

$$A_{\eta\hat{}k} = F(x_{1,\eta_1}, A_{\eta|1}, x_{2,\eta_2}, A_{\eta|2}, \dots, x_{m,\eta_m}, A_\eta, x_{m+1,k}),$$

where for each i, η_i is the ith element of η and $\eta|i$ is the sequence $(\eta_1, \dots, \eta_i)$.

Next, for each η, define $B_\eta = \bigcap_k A_{\eta\hat{}k}$. Assume that $A \not\subseteq \bigcup_\eta B_\eta$, and let $a \in A \setminus \bigcup_\eta B_\eta$. Choose inductively k_1 such that $a \notin A_{k_1}$, k_2 such that

$a \notin A_{(k_1,k_2)}$, etc. Then the play $(x_{1,k_1}, A_{k_1}, x_{2,k_2}, A_{(k_1,k_2)}, \dots)$ is according to the strategy F and lost by **II**, a contradiction. Consequently, $A \subseteq \bigcup_\eta B_\eta$.

For each m and each $\eta \in \mathbb{N}^m$, $B_\eta = \bigcap_k A_{\eta\hat{\ }k} \in \mathcal{A}(\wedge\ D \cap U_m)$. Consequently, $C_m = \bigcup_{\eta\in\mathbb{N}^m} B_\eta \in \mathcal{A}(\wedge\ D \cap U_m)$ too, and $A \subseteq \bigcup_m C_m$ as required. $\square$

Corollary 2.10. *Assume that* $\mathbf{II} \uparrow \mathcal{A}_{\mathsf{G}}$ *holds for* A. *Then: For each countable dense* $D \subseteq X$, *there exist elements*

$$A_n \in \mathcal{A}(\wedge\ D \cap B(x_n, \epsilon_n)),$$

where each $x_n \in X$ *and each* $\epsilon_n > 0$, *such that* $A \subseteq \bigcup_n A_n$. $\square$

Problem 2.11. *Is the property in Corollary 2.10 equivalent to* $\mathbf{II} \uparrow \mathcal{A}_{\mathsf{G}}$, *or at least implies* $\mathbf{I} \not\uparrow \mathcal{A}_{\mathsf{G}}$*?*

3. Selection hypotheses

Definition 3.1. $\mathbb{A}$ is the collection of Borel sets $A \subseteq X$ such that: For each sequence $\{D_n\}_{n\in\mathbb{N}}$ of dense subsets of X, there exist elements $x_n \in D_n$ and $A_n \in \mathcal{A}(x_n)$, $n \in \mathbb{N}$, such that $A \subseteq \bigcup_n A_n$. $\mathbb{A}_{\mathsf{G}}$ is the corresponding game, played as follows:

$$\begin{array}{lccccc}
\mathbf{I}: & D_1 \subseteq X & & D_2 \subseteq X & & \dots \\
 & & \searrow \quad \nearrow & & \searrow & \\
\mathbf{II}: & & \begin{array}{c} x_1 \in D_1, \\ A_1 \in \mathcal{A}(x_1) \end{array} & & \begin{array}{c} x_2 \in D_2, \\ A_2 \in \mathcal{A}(x_2) \end{array} & \dots
\end{array}$$

where each D_n is dense in X, and **II** wins the game if $A \subseteq \bigcup_n A_n$; otherwise **I** wins.

The appealing property in the game $\mathbb{A}_{\mathsf{G}}$ is that, unlike the case in the game $\mathcal{A}_{\mathsf{G}}$, there is no commitment of **I** which has to be verified "at the end" of the play.

Proposition 3.2. $\mathbb{A} = \mathcal{A}$.

Proof. ($\subseteq$) Assume that $A \in \mathbb{A}$, and let $D = \{x_n\}_{n\in\mathbb{N}}$ be dense in X. For each n, take $D_n = D$ and apply $\mathbb{A}$. Then there are $y_n \in D$ and $A_n \in \mathcal{A}(y_n)$, $n \in \mathbb{N}$, such that $A \subseteq \bigcup_n A_n$. As each $\mathcal{A}(x_n)$ is σ-additive, we may assume that no x_n appears more than once in the sequence $\{y_n\}_{n\in\mathbb{N}}$. Thus, $A \in \mathcal{A}$.

($\supseteq$) Assume that $A \in \mathcal{A}$, and let $\{D_n\}_{n\in\mathbb{N}}$ be a sequence of dense subsets of X. For each n choose $x_n \in D_n$ such that $D = \{x_n\}_{n\in\mathbb{N}}$ is dense in X (to do that, fix a countable base $\{U_n\}_{n\in\mathbb{N}}$ for the topology of X, and for each n pick $x_n \in U_n \cap D_n$). By $\mathcal{A}$, there exist sets $A_n \in \mathcal{A}(x_n)$ such that $A \subseteq \bigcup_n A_n$. This shows that $A \in \mathbb{A}$. $\square$

A simple modification of the last proof gives the following.

Theorem 3.3. $\mathbf{I}\uparrow\mathbb{A}_{\mathsf{G}}$ *if, and only if,* $\mathbf{I}\uparrow\mathcal{A}_{\mathsf{G}}$.

Proof. ($\Rightarrow$) Let F be a winning strategy for $\mathbf{I}$ in the game $\mathbf{I}\uparrow\mathbb{A}_{\mathsf{G}}$ on A. Define a strategy for $\mathbf{I}$ in the game $\mathbf{I}\uparrow\mathcal{A}_{\mathsf{G}}$ as follows. Fix a countable base $\{U_n\}_{n\in\mathbb{N}}$ for the topology of X. In the first inning, $\mathbf{I}$ plays any $x_1\in U_1\cap D_1$ where D_1 is $\mathbf{I}$'s first move according to the strategy F. Assume that the first n moves where $(x_1,A_1,\dots,x_{n-1},A_{n-1})$. Let $D_n=F(D_1,(x_1,A_1),\dots,D_{n-1},(x_{n-1},A_{n-1}))$. Then $\mathbf{I}$ plays any $x_n\in U_n\cap D_n$. For each play $(x_1,A_1,x_2,A_2,\dots)$ according to this strategy, $\{x_n\}_{n\in\mathbb{N}}$ is dense in X, and since $(D_1,(x_1,A_1),D_2,(x_2,A_2),\dots)$ is a play in the game $\mathbb{A}_{\mathsf{G}}$ according to the strategy F, $A\not\subseteq\bigcup_n A_n$.

($\Leftarrow$) Let F be a winning strategy for $\mathbf{I}$ in the game $\mathbf{I}\uparrow\mathcal{A}_{\mathsf{G}}$ on A. Define a strategy for $\mathbf{I}$ in the game $\mathbf{I}\uparrow\mathbb{A}_{\mathsf{G}}$ as follows. $\mathbf{I}$'s first move is D_1, the set of all points x which are possible moves of $\mathbf{I}$ at some inning according to its strategy F. Obviously, D_1 is dense. In the nth inning, we are given $(D_1,(x_1,A_1),\dots,D_{n-1},(x_{n-1},A_{n-1}))$, such that there is a sequence of moves $(y_1,B_1,y_2,B_2,\dots,y_{k_n})$ according to the strategy F, with $y_{k_n}=x_{n-1}$. Then $\mathbf{I}$ plays D_n, the set of all points x which are possible moves of $\mathbf{I}$ at some future inning, in a play according to the strategy F whose first moves are $(y_1,B_1,y_2,B_2,\dots,y_{k_n}=x_{n-1},A_{n-1})$. $(y_1,B_1,y_2,B_2,\dots)$ is a play according to the strategy F, and therefore $A\not\subseteq\bigcup_n B_n\supseteq\bigcup_n A_n$, so that $A\not\subseteq\bigcup_n A_n$. $\square$

The following is immediate.

Lemma 3.4. *If* $\mathbf{II}\uparrow\mathcal{A}_{\mathsf{G}}$, *then* $\mathbf{II}\uparrow\mathbb{A}_{\mathsf{G}}$. $\square$

Problem 3.5. *Is it true that* $\mathbf{II}\uparrow\mathbb{A}_{\mathsf{G}}$ *if, and only if,* $\mathbf{II}\uparrow\mathcal{A}_{\mathsf{G}}$*?*

By Theorem 3.3 and Proposition 3.2, Conjecture 2.4 can be reformulated as follows.

Conjecture 3.6. *For a Borel set* $A\subseteq X$*:* $\mathbf{I}\not\uparrow\mathbb{A}_{\mathsf{G}}$ *if, and only if,* $A\in\mathbb{A}$.

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